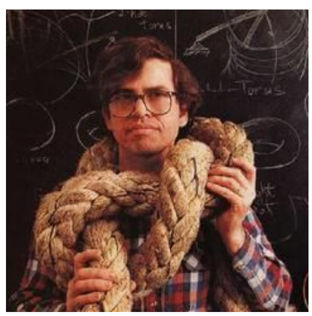
Worksheet 12 **Name:** ____________________
Week 12: 2024-11-12
Math 294: Fall 2024
Instructor: Theo Sandstrom

Knot theory!

William Thurston (1946-2012) was an American mathematician who made pioneering advances in the field of low-dimensional topology. Broadly, the mathematical field of topology studies "flexible" geometry; it has been described as the "geometry of a rubber sheet." In studying shapes made out of infinitely elastic rubber, the notions of distance and angle that underpin classical geometry become meaningless, but other tools can still be used to distinguish shapes. As an example, no amount of stretching or bending can deform a donut into a sphere. While topological tools can be used to study shapes in any number of dimensions, many unique phenomena occur only in low dimensions ($\leqslant 4$). Thurston's career was focused on 3-dimensional geometry, with a particular focus on hyperbolic geometry; his 1982 geometrization conjecture, describing all three-dimensional shapes in terms of 8 basic geometries, formed the basis for an entire field of research, until it was proven by Perelmen in 2002.

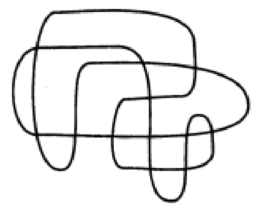
I've found Thurston's article On Proof and Progress in Mathematics to be quite an interesting philosophical take on the goals of mathematics.



Problem 1. Show that there are no two-crossing nontrivial knots.

Problem 2. Show that any knot admits a projection with over 1000 crossings.

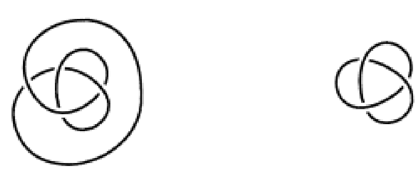
Problem 3. A knot is called *alternating* if it admits a projection that has crossings alternating between over and under as one travels around the knot in a fixed direction. Choose crossings at each vertex to make the below knot alternating.



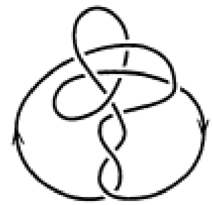
Problem 4 (Challenge). Show that by changing the crossings from over to under (or vice versa), any projection of a knot can be made into an alternating projection (of a different knot). It might not be too hard to come up with a procedure; the tricky part is to prove that your procedure always works.

Problem 5 (Challenge). Show that by changing the crossings from over to under (or vice versa), any projection of a knot can be made into a projection of the trivial knot.

Problem 6. Show that the two projections below represent the same knot by finding a series of Reidemeister moves from one to the other.



Problem 7. Compute the linking number of the below link:



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Problem 8. According to our definition of linking number, we should start by choosing a projection of our link. Prove that it doesn't matter which projection we pick; that is, show that linking number is a *link invariant.* (Hint: show that Reidemeister moves do not change linking number.)

Problem 9. A *strand* in a projection of a link is a piece of the link that goes from one undercrossing to another with only overcrossings in between. A projection of a knot or link is called *tricolorable* if each of the strands can be colored one of three colors so that, at each crossing, either three different colors come together, or all the same color comes together. Moreover, we require that at least two of the colors be used.

Is the trefoil knot tricolorable?

Problem 10. Prove that tricolorability is a knot invariant; that is, it does not depend on which projection of the knot that you pick. (Hint: show that Reidemeister moves do not change tricolorability; that is, if a projection is tricolorable, then it remains tricolorable after a Reidemeister move.)

Problem 11. Show that the trefoil knot and the unknot are distinct knots.