Worksheet 10 **Name:** ____________________
Week 10: 2024-10-29
Math 294: Fall 2024
Instructor: Theo Sandstrom

More groups!

Maryam Mirzakhani (1977-2017) was an Iranian mathematician, and the first woman to receive the Fields medal (the mathematical equivalent of a Nobel prize). The award committee cited her contributions to the understanding of the "dynamics and geometry of Riemann surfaces and their moduli spaces"—in simpler terms, studying the geometric relationships between all possible surfaces that locally behave like the complex plane. Raised in Tehran, her mathematical talent was evident from a young age; she earned a perfect score on the International Mathematical Olympiad in Toronto. Her doctoral thesis at Harvard was published in the three top math research journals: *Annals of mathematics*, *Inventiones*, and *The Journal of the AMS*. She taught at Princeton and then Stanford, where she continued making fundamental contributions to Teichmüller theory until her death due to breast caner at age 40.

Problem 1. Suppose that G is a finite group. Show that every element $g \in G$ has finite order.

Problem 2. Suppose that G is a cyclic group. Show that G is abelian.

Problem 3. Let G be a group, and $g, h \in G$ any two elements. Show that $(gh)^{-1} = h^{-1}g^{-1}$. (Note: This is sometimes called the *shoes–socks principle*. Can you think of why?)

Problem 4. Show that a group G is abelian if and only if $(gh)^{-1} = g^{-1}h^{-1}$ for every $g, h \in G$.

Problem 5. Draw a Cayley graph for the following groups: $\mathbb{Z}$, $\mathbb{Z}_n$, D_3, D_4. (Note: you will have to pick a set of generators for each group.)

Problem 6. If G and H are groups, a function $\varphi : G \to H$ is called a *group homomorphism* if, for every elements $g, g' \in G$, $\varphi(gg') = \varphi(g)\varphi(g')$. Are the following homomorphisms?

(a) $f : D_3 \to D_3$ given by $f(\sigma) = \sigma^{-1}$. (Recall: D_3 is the group of rigid symmetries of the triangle.)

(b) Considering $(\mathbb{Z}, +)$ as a group under addition, $g : \mathbb{Z} \to \mathbb{Z}$ given by $g(n) = 2n$.

(c) Again considering the integers under addition, $h : \mathbb{Z} \to \mathbb{Z}$ given by $h(n) = n + 1$.

(d) Letting $G = (\mathbb{R}, +)$ and $H = (\mathbb{R}_{>0}, \cdot)$, $\exp : G \to H$ given by $\exp(x) = e^x$.

Problem 7. If $\varphi : G \to H$ is a homomorphism of groups, show that $\varphi(e_G) = e_H$ and that, for each $g \in G$, $\varphi(g^{-1}) = \varphi(g)^{-1}$.

Problem 8.

(a) Let H be a group and $h \in H$ any element. Show that if $h^n = e$, then the order of h divides n. (Hint: use the division algorithm.)

(b) If $\varphi : G \to H$ is a homomorphism of groups, and $g \in G$ is an element of finite order. Show the order of $\varphi(g)$ divides the order of g.

Problem 9. Recall that, if $f : S \to T$ is a function between two sets, then it is injective if and only if it admits a left inverse. Compare with the following: let $\varphi : \mathbb{Z}_2 \to \mathbb{Z}_4$ be defined by $\varphi([0]_2) = [0]_4$ and $\varphi([1]_2) = [2]_4$.

(a) Is φ a group homomorphism?

(b) Is φ injective?

(c) Show that there does not exist any group homomorphism $\psi : T \to S$ so that $\psi \circ \varphi = \mathrm{id}_S$.

Problem 10 (Challenge). Can you find a *geometric* explanation for the fact that the group of rotations of the cube is isomorphic to the group of rigid symmetries of the tetrahedron?