Worksheet 7 **Name:** ____________________

Week 7: 2024-10-08

Math 294: Fall 2024

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We've seen sets; that's about half of math. Now time for the other half: functions! As always, work together, and ask lots of questions, both of each other, and of me!

Karen Uhlenbeck Is First Woman to Win Abel Prize for Mathematics by Kenneth Chang, March 19, 2019 (excerpt from NY Times article).

For the first time, one of the top prizes in mathematics has been given to a woman.

On Tuesday, the Norwegian Academy of Science and Letters announced it has awarded this year's Abel Prize—an award modeled on the Nobel Prizes—to Karen Uhlenbeck, an emeritus professor at the University of Texas at Austin. The award cites "the fundamental impact of her work on analysis, geometry and mathematical physics."

One of Dr. Uhlenbeck's advances in essence described the complex shapes of soap films not in a bubble bath but in abstract, high-dimensional curved spaces. In later work, she helped put a rigorous mathematical underpinning to techniques widely used by physicists in quantum field theory to describe fundamental interactions between particles and forces.

In the process, she helped pioneer a field known as geometric analysis, and she developed techniques now commonly used by many mathematicians.

"She did things nobody thought about doing," said Sun-Yung Alice Chang, a mathematician at Princeton University who served on the five-member prize committee, "and after she did, she laid the foundations of a branch of mathematics."

The Abel, named after the Norwegian mathematician Niels Henrik Abel, is set up more like the Nobels. Since 2003, it has been given out annually to highlight important advances in mathematics. The previous 19 laureates—in three years, the prize was split between two mathematicians—were men, including Andrew J. Wiles, who proved Fermat's last theorem and is now at the University of Oxford; Peter D. Lax of New York University; and John F. Nash Jr., whose life was portrayed in the movie "A Beautiful Mind."

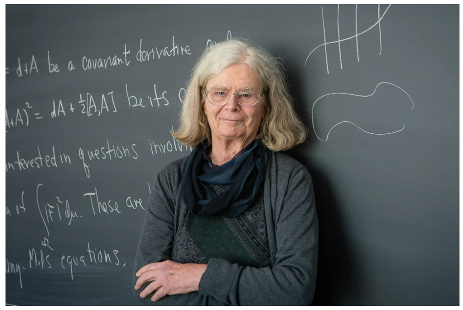
In 1990, she became the second woman to give one of the highlighted plenary talks at the International Congress of Mathematicians, a quadrennial meeting. At each congress, there are 10 to 20 plenary talks, but for decades, all of the speakers had been men. (Emmy Noether, a prominent German mathematician, was the first woman to give a plenary talk, in 1932.)

"That was almost more unnerving" than being the first woman to receive an Abel, Dr. Uhlenbeck said.

Dr. Uhlenbeck said she recognized that she was a role model for women who followed her in mathematics.

"Looking back now I realize that I was very lucky," she said. "I was in the forefront of a generation of women who actually could get real jobs in academia."

But she also noted: “I certainly very much felt I was a woman throughout my career. That is, I never felt like one of the guys.”



Problem 1. We have already encountered the *existential quantifier* $\exists$, which can be used in a proposition like “$\exists x \in X$ such that $P(X)$”. A slight strengthening is the *unique existential quantifier*: the proposition “$\exists!\, x \in X$ such that $P(X)$” says two things: first that there exists an x satisfying P, **and** that there is exactly one such x. How can you express this meaning *without* using the special $\exists!$ quantifier? What steps are required to prove such a proposition?

Problem 2. Let X and Y be sets. A *function* f from X to Y (denoted $f : X \to Y$) is a specification of outputs $f(x) \in Y$ for each input $x \in X$, so that

$$\forall x \in X, \exists!\, y \in Y, y = f(x).$$

The set X is called the *domain* while the set Y is called the *codomain.* A well-defined function must satisfy the following properties:

(a) **Totality:** Every value $x \in X$ is assigned a value $f(x)$.

(b) **Existence:** For each $x \in X$, the value $f(x)$ should actually exist as an element of the codomain.

(c) **Uniqueness:** For each $x \in X$, there should be only one value assigned as $f(x)$.

Using whatever notation/description you prefer, give three examples of “non-functions” that fail to satisfy each of the above criteria.

Problem 3. Prove that there exists a unique real number a for which the equation $x^2 + a^2 = 0$ has a real solution for x

Problem 4. For each of the following "specifications", determine if they are well-defined functions. If not, identify the property that fails.

(a) $f : \mathbb{Q} \to \mathbb{Q}$ given by $f(x) = (x+1)^{-1}$.

(b) $g : \mathbb{Z}_{\geqslant 0} \to \mathbb{Q}$ given by $g(x) = (x+1)^{-1}$.

(c) $h : \mathbb{N} \to \mathbb{N}$ given by $h(x) = (x+1)^{-1}$.

[Note: even though f, g, and h are all defined by identical formulae, they are not the same mathematical objects because their (co-)domains are different.]

Problem 5. Does the specification

$$f(x) = \begin{cases} x+1 & \text{if } x \geqslant 0, \\ 0 & \text{if } x \leqslant 0, \end{cases}$$

determine a well-defined function? If not, why not? Can this be remedied easily?

Problem 6. Give a condition on the sets X and Y that will ensure that the specification of a well-defined function $i : X \cup Y \to \{0, 1\}$ given by

$$i(x) = \begin{cases} 1 & \text{if } x \in X, \\ 0 & \text{if } x \in Y. \end{cases}$$

Problem 7. Let $f : X \to Y$ be a function, and let $U \subseteq X$. The *image* of U under f, denoted $f(U)$, is the subset of Y given by $f(U) = \{f(x) \mid x \in U\}$. Prove that we could equivalently define $f(U) = \{y \in Y \mid \exists x \in X \text{ so that } y = f(x)\}$.

Problem 8. Let $f : X \to Y$ be a function, and let $V \subseteq Y$. The *preimage* of V under f, denoted $f^{-1}(V)$, is the subset of X given by $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$.

(a) Is it true that $f(X) = Y$?

(b) Is it true that $f^{-1}(Y) = X$? When does there exist a smaller set $V \subseteq Y$ so that $f^{-1}(V) = X$?

Problem 9. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sqrt{1 + x^2}$. Is this a well-defined function? If so, describe its image $f(X)$.

Problem 10. Let $f : X \to Y$ be a function, and let $U, V \subseteq X$. Is it true that $F(U) \cup f(V) = f(U \cup V)$? (Provide proof.)

Problem 11. Let $f : X \to Y$ be a function, and let $U, V \subseteq Y$. Is it true that $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$? (Provide proof.)

Problem 12. Let $f : X \to Y$ be a function, and let $U, V \subseteq X$. Is it true that $f(U) \cap f(V) = f(U \cap V)$? (Provide proof.)

Problem 13. Let $f : X \to Y$ be a function, and let $U \subseteq X$ and $V \subseteq Y$. Is it true that $f^{-1}(f(U)) = U$? If not, which (if any) containment *is* true?

Similarly, is it true that $f(f^{-1}(V)) = V$? Which containment is true in this case?