Worksheet 5 **Name:** ______________________________
Week 5: 2024-09-24
Math 294: Fall 2024
Instructor: Theo Sandstrom

This week, the goal is to work some slightly more formal set-theoretic notions into our proofs. The ideas of sets and functions is central to the development of lots of math, but can take some time to get used to! Practice with these problems, or the problems from your 215 section. Work together! And again, please come to office hours if you have any questions, ideas, things to clarify, or just want to talk math!

"We should take great care not to accept as true such properties of the numbers which we have discovered by observation and which are supported by [heuristic] alone. Indeed, we should use such discovery as an opportunity to investigate more exactly the properties discovered and to prove or disprove them; in both cases we may learn something useful." —Leonhard Euler (1707-1783)

Euler was born in Basel, Switzerland, and he is widely regarded as one of the greatest mathematicians of all time. He studied under Johann Bernouilli and finished university at age 15, earning a position at the Imperial Russian Academy of Sciences by age 20. He had a love for music, language, philosophy, science, and mathematics, and wrote on all of these subjects.

Throughout his life, Euler produced over 25,000 pages of mathematics in virtually every field; Wikipedia lists over 90 mathematical concepts named after him. One of the first results that rocketed him to mathematical fame was his proof that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6},$$

solving the so-called *Basel problem.*

At age 64, a failed procedure left Euler blind, to which he said *"Now I will have less distraction"*. His mathematical productivity continued to increase during this period; in 1775, he averaged one new research paper *every week.* Besides being an unparalleled researcher, Euler was also an excellent teacher and expositor, writing numerous influential books which still influence modern pedagogical approaches. He introduced the function notation $f(x)$, summation notation using $\sum$, the symbols for the constants π, e, and i, and much more.

"Read Euler, read Euler, he is the master of us all." — Pierre-Simon Laplace

Problem 1. Write the following sets using "set builder" notation:

(a) The set of all odd integers (you may assume that $\mathbb{Z}$ is already defined).

(b) The set of all positive real numbers (you may assume that $\mathbb{R}$ is already defined).

(c) The set of rational numbers with denominator 2 (when written as reduced fractions).

Problem 2. Give a description of the following sets in plain/natural English:

(a) $\{n \in \mathbb{Z} \mid \nexists k \in \mathbb{Z} \text{ so that } n = 3k\}$.

(b) $\{k \in \mathbb{Z} \mid 1 \leqslant k \leqslant n\}$, where n is some fixed integer.

(c) $\{x \in \mathbb{R} \mid x^2 \leqslant 1\}$.

Problem 3. Let X and Y be two sets, and suppose that $P(a)$ denotes the statement '$a \in X$' while $Q(a)$ denotes the statement '$a \in Y$'. Write a logical formula representing the meaning of each of the following:

(a) $a \notin X$

(b) $a \in X \cap Y$

(c) $a \in X \cup Y$

(d) $a \in X \backslash Y$

(e) $X \subseteq Y$

Problem 4. Recall de Morgan's laws for propositional logic (these describe the negation of conjunction ($\wedge$) and disjunction ($\vee$)). Based on the previous exercise, can you find a similar law for set arithmetic? Namely, give an alternative way to write the following:

(a) $A \backslash (X \cap Y)$

(b) $A \backslash (X \cup Y)$

Problem 5. Given real numbers $x < y$, recall that (x, y) denotes the *open* interval from x to y, while $[x, y]$ denotes the *closed* interval.

(a) Express these intervals using "set builder" notation.

(b) If $a < b < c < d$, prove that $[b, c] \subseteq (a, d)$.

Problem 6. Explain in plain English the meaning of the non-containment $X \nsubseteq Y$. How would you prove such a statement? If you are granted the assumption that $X \nsubseteq Y$, what information does this give you, and how might you write this down in a proof?

Problem 7. Show that $\{x \in \mathbb{R} \mid x^2 < x\} = (0, 1)$.

Problem 8. Challenge Consider a $2 \times n$ rectangle, for n some positive integer.

(a) Show that the rectangle can always be tiled by 1×2 blocks.

(b) How many different ways can it be tiled by 1×2 blocks?

(c) How many different ways can a $3 \times n$ rectangle be tiled by 1×2 blocks?