Worksheet 4 **Name:** ____________________

Week 4: 2024-09-17

Math 294: Fall 2024

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Problems for Tuesday and Thursday. Play around with the math, and work together! And please come to office hours if you have any questions, ideas, things to clarify, or just want to talk math! Problem 8 is one of my favorite math puzzles!

In January of 1913, Cambridge mathematician G. H. Hardy (1877-1947) received an unusual piece of mail, covered in stamps and posted from the other side of the world. The sender was an unheard of Indian clerk named Srinivasa Ramanujan, and Hardy initially wrote the letter off as just another piece of mathematical nonsense from some fraud claiming to have cracked open an unsolved problem. When he saw the following line, his opinion immediately changed:

$$\frac{e^{-2\pi/5}}{1+\frac{e^{-2\pi}}{1+\frac{e^{-4\pi}}{1+\frac{e^{-6\pi}}{1+\ddots}}}} = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{1+\sqrt{5}}{2}.$$

The letter contained pages upon pages of similar equations, with little explanation, and not a word of proof. Hardy, intrigued, set to proving the claims, but after several days of work, he wrote back to Ramanujan, defeated. He later said:

"I had never seen anything in the least like this before. A single look at them is enough to show they could only be written down by a mathematician of the highest class. They must be true because if not, no one would have the imagination to invent them."

Srinivasa Ramanujan (1887-1920) was born in Tamil Nadu in the south of India during the British Raj. As a child and teen, living in extreme poverty and on the brink of starvation, he showed a remarkable aptitude for mathematics despite receiving little formal schooling. While briefly attending college on scholarship, he proved such results as

$$3 = \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{\ldots}}}}$$

in the *Journal of the Indian Mathematical Society.*

When Hardy received his letter, he insisted that Ramanujan come to work with him at Cambridge. Initially refusing to leave his family, Ramanujan was only convinced when his mother purportedly had a vision that the family goddess wanted him to study with hardy. In England, his brilliance was immediately recognized; he would often state remarkable theorems without proof, and Hardy and the best mathematicians at Cambridge would struggle to prove them. He attributed these theorems to divine inspiration, saying *"an equation has no meaning for me unless it expresses a thought of God."* Recalling his work with Ramanujan, Hardy later said

"Every positive integer was one of Ramnujan's personal friends... I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the

number seemed to me a rather dull one, and that I hoped it was not an unfavorable omen. 'No!' he replied, 'It is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways."'

Problem 1. Let n be an integer. Show that if $7n + 5$ is odd, then n is even. (Try to give two proofs: one by contrapositive, one direct.)

Problem 2. Let x and y be integers. Prove that $3 \mid xy$ if and only if $3 \mid x$ or $3 \mid y$.

Problem 3. Prove that $\sqrt{3}$ is irrational.

Problem 4. Rewrite the previous proof to show that $\sqrt{4}$ is irrational. Is it still correct?

Problem 5. Let x be a real number. Prove that if $2x^3 - x^2 + 4x - 1 \geqslant 0$, then $x \geqslant 0$.

Problem 6. Let x be a rational number and a be an irrational number. Show that $a + b$ is irrational. If $a \neq 0$, show that ab is irrational.

Problem 7 (Challenge). Let x and y be *positive* real numbers. Prove that if $x \neq y$, then $\frac{x}{y} + \frac{y}{x} > 2$.

Problem 8. Challenge You are presented with a standard 8×8 checkerboard, and a bag full of 2×1 dominoes. You may rotate the dominoes in increments of 90°, and you may place them on the checkerboard so that they cover exactly two squares.

(a) Can you arrange dominoes to cover the entire board?

(b) If the single square in the upper-right-hand corner is removed from the board, can you still tile the board with dominoes?

(c) What if, in addition, the single square in the bottom-left-hand corner is removed from the board?