Worksheet 3 **Name:** ______________________________
Week 3: 2024-09-10
Math 294: Fall 2024
Instructor: Theo Sandstrom

These problems are to be completed in class on Tuesday and Thursday. Work in groups! And come to office hours if you have further questions! They're great!

"My methods are really methods of working and thinking; this is why they have crept in everywhere anonymously." —Amalie Emmy Noether (1882-1935)

Emmy Noether was a German mathematician who made foundational contributions to abstract algebra and physics despite the profound professional barriers to women at the time. Her career began when David Hilbert and Felix Klein, noticing her ingenuity in the field of invariant theory, invited her to the University of Göttingen in 1915. Despite their high praise (Hilbert and Klein were enormously influential mathematicians), the university refused outright to offer any position, saying "what will our soldiers think when they return to the university and find that they are required to learn at the feet of a woman?"

Noether worked as a lecturer at Göttingen without pay for several years; eventually the university started to pay her, but she was never granted the title of full professor. Her mathematical work ranged from ring and field theory to algebraic topology, and laid the foundation for much of modern commutative algebra and algebraic geometry.

Problem 1. Convert the following statements from English into formal logic. (Note that they are not necessarily true.)

(a) There is an integer that is divisible by every integer. (Note: one can write $k \mid n$ to mean "k divides n" or "n is divisible by k".)

(b) There is no greatest integer.

(c) There is no greatest odd integer.

Problem 2. Translate the following logical formula to plain English, where x represents a real number. Use as few variables as possible.

$$\exists m \in \mathbb{Z}, \exists n \in \mathbb{Z}, (n \neq 0 \wedge nx = m).$$

Problem 3 (Repeat). Let x and y be real numbers. Prove that if both $x + y$ and $x - y$ are rational, then x and y are rational.

Problem 4. Prove that, for all $x, y \in \mathbb{Q}$, if $x < y$ then there exists some $z \in \mathbb{Q}$ so that $x < z < y$.

Problem 5. Recall De Morgan's laws for quantifiers, which state:

$$\neg(\exists x : P(x)) \iff \forall x : (\neg P(x)), \qquad \neg(\forall x : P(x)) \iff \exists x : (\neg P(x))$$

Problem 6. Prove that, if an integer n is *not* divisible by 3, then there exists an integer k so that $n^2 = 3k + 1$.

Prove that, if r, s, t are *all* not divisible by 3, then $r^2 + s^2 + t^2$ is divisible by 3.

Problem 7. Explain the difference between the following statements:

(a) $\exists x \in X$ such that $\forall y \in Y$, $P(x, y)$.

(b) $\forall y \in Y$, $\exists x \in X$ such that $P(x, y)$.

Problem 8. Prove that, for any real number $\delta > 0$ and any $M \in \mathbb{R}$, there exists some $n \in \mathbb{Z}$ so that $n\delta > M$.

Problem 9. Prove that, for any real number $t > 1$ and any $M \in \mathbb{R}$, there exists some $n \in \mathbb{Z}$ so that $t^n > M$.

Problem 10. Prove that, for any $0 < t < 1$ and any $\varepsilon > 0$, there exists some $n \in \mathbb{Z}$ so that $t^n < \epsilon$.